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THE FREE OSCILLATIONS OF A VISCOUS TWO-LAYER FLUID IN A CLOSED VESSEL*

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The spatial, infinitely small oscillations of a viscous two-layer heavy fluid in a vessel of arbitrary shape are investigated. The Reynolds number is assumed to be large (low viscosity) which enables one to use the ideas of boundary-layer theory and the Krylov-Bogolyubov averaging method. Boundary-layer theory is used as in /1/ and /2/ in which linear problems on the oscillations of a compressible medium with a low viscosity were solved. Approximation formulae are derived for the velocity components of the fluid and, also, for the damping coefficient and the correction to the frequency of the free oscillations of an ideal fluid. Like the analogous quantities in /1/, these quantities are expressed in terms of the eigenvalues and eigenfunctions of the corresponding problem on the oscillations of an ideal fluid. It is found that the damping coefficient and the correction to the frequency, even in the first approximation with respect to the small parameter, also depend, as in the two-dimensional case, on the loss of energy in the boundary layers close to the boundary of separation of the two-layer fluid. In the case of a homogeneous fluid in an open vessel, the energy losses close to the free surface of the fluid are asymptotically small (see /1/) compared with the losses close to the walls of the vessel. Expressions for the damping coefficient and the correction to the frequency of the vibrations for vessels with the form of a rectangular parallelepiped and a circular cylinder are given as examples.

The free vibrations of a homogeneous viscous heavy fluid in a vessel of arbitrary shape were investigated in /1/. The two dimensional oscillations of a viscous two-layer fluid were considered in /3-5/.

1. Initial equations. Let us consider the problem of the free, infinitely small, oscillations of a two-layer, heavy, viscous, compressible fluid which completely fills a closed vessel. We assign the index 1 to all quantities referring to the upper layer of the lighter fluid and the index 2 to all quantities referring to the lower layer. In the domains D_m occupied by the fluid, the velocities $\mathbf{U}_m$ $(m = 1, 2)$ of the fluid particles and the pressure, P_m, must satisfy the Navier-Stokes and continuity equations. The velocities $\mathbf{U}_m$ must be equal to zero on the walls of the vessel, S_m. The velocities $\mathbf{U}_1$ and $\mathbf{U}_2$, as well as the normal and tangential stresses, must remain continuous on crossing the boundary of separation of the fluid layers, Σ. By virtue of the assumption regarding the infinite smallness of the motions, we assume that the domains D_m and the surface Σ are always the same as in the equilibrium state of the fluid. We also assume that the partial derivative of the elevation of the boundary of separation H, with respect to time is equal to the vertical components of the velocities of the upper and lower fluids.

$$\begin{gathered}
\partial \mathbf{U}_m/\partial t = -\rho_m^{-1}\nabla P_m - g\mathbf{k} + \nu_m \Delta \mathbf{U}, \ \operatorname{div} \mathbf{U}_m = 0 \quad \text{in} \quad D_m \\
\mathbf{U}_m = 0 \quad \text{on} \quad S_m \\
-P_1 - \frac{\partial P_1}{\partial z} H + 2\nu_1\rho_1 \frac{\partial U_{1z}}{\partial z} = -P_2 - \frac{\partial P_2}{\partial z} H + 2\nu_2\rho_2 \frac{\partial U_{2z}}{\partial z} \\
\nu_1\rho_1 \left(\frac{\partial U_{1\xi}}{\partial z} + \frac{\partial U_{1z}}{\partial \xi}\right) = \nu_2\rho_2 \left(\frac{\partial U_{2\xi}}{\partial z} + \frac{\partial U_{2z}}{\partial \xi}\right), \quad \xi = x, y \\
\mathbf{U}_1 = \mathbf{U}_2, \ U_{mz} = \partial H(x, y, t)/\partial t \quad \text{on} \quad \Sigma, \ m = 1, 2
\end{gathered} \tag{1.1}$$

Here ρ_m is the density, ν_m are the kinematic viscosities of the fluids, and the rectangular Cartesian coordinate system xyz is chosen such that the xy plane coincides with the Σ plane while the z-axis and the unit vector $\mathbf{k}$ are directed vertically upwards.

Let us now introduce dimensionless variables by adopting the characteristic size of the vessel, d, as the unit of length and $T_0 = 1/\omega_0$ as the unit of time, where ω_0 is the smallest characteristic vibrational frequency of the ideal fluid:

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$$x = dx^*,\ y = dy^*,\ \ z = dz^*,\ t = t^*/\omega_0,\ \rho_1 = \rho\rho_2,\ \nu_1 = \nu\nu_2 \tag{1.2}$$

$$\mathbf{U}_m = d\omega_0\mathbf{u}_m,\ P_m = -g\rho_m z + d^2\omega_0^2\rho_m p_m,\ H = d\eta$$

The velocity vectors of the fluids, $\mathbf{U}_m$ are represented in the form of the sum of potential and vortex components

$$\mathbf{u}_m = -\nabla\varphi_m + \mathbf{v}_m \tag{1.3}$$

and, here and below while omitting the asterisk, we assume that $p_m = \partial\varphi_m/\partial t$ and the functions φ_m and $\mathbf{v}_m$ satisfy the equations

$$\Delta\varphi_m = 0;\ \partial\mathbf{v}_m/\partial t = R^{-1}\mu_m\Delta\mathbf{v}_m,\ \operatorname{div}\mathbf{v}_m = 0 \tag{1.4}$$

in D_m. ($R = \omega_0 d^2/\nu_2$, $\mu_m = \nu\delta_{1m} + \delta_{2m}$ and δ_{kl} is the Kronecker delta).

The differential equations of the problem will then be satisfied in the domains D_m. The boundary conditions for φ_m and $\mathbf{v}_m$ are written as:

$$\nabla\varphi_m = \mathbf{v}_m \quad \text{on} \quad S_m \tag{1.5}$$

$$\nabla\varphi_2 - \nabla\varphi_1 = \mathbf{v}_2 - \mathbf{v}_1$$

$$\frac{\partial\varphi_2}{\partial z} - \rho\frac{\partial\varphi_1}{\partial z} + F\frac{\partial^2}{\partial t^2}(\varphi_2 - \rho\varphi_1) = v_{2z} - \rho v_{1z} +$$

$$\frac{2F}{R}\frac{\partial}{\partial t}\left(\frac{\partial v_{2z}}{\partial z} - \rho\nu\frac{\partial v_{1z}}{\partial z} + \rho\nu\frac{\partial^2\varphi_1}{\partial z^2} - \frac{\partial^2\varphi_2}{\partial z^2}\right) \quad \left(F = \frac{\omega_0^2 d}{g}\right)$$

$$\frac{1}{\sqrt{R}}\left[2\rho\nu\frac{\partial^2\varphi_1}{\partial\xi\partial z} - 2\frac{\partial^2\varphi_2}{\partial\xi\partial z} - \rho\nu\left(\frac{\partial v_{1\xi}}{\partial z} + \frac{\partial v_{1z}}{\partial\xi}\right) + \frac{\partial v_{2\xi}}{\partial z} + \frac{\partial v_{2z}}{\partial\xi}\right] = 0$$

$$\xi = x, y \quad \text{on} \quad \Sigma$$

We subsequently assume that $R \gg 1$.

2. ***General solution scheme.*** The system of Eqs.(1.4) is a singularly perturbed system since it contains a small parameter $1/R$ accompanying the highest derivative. We seek the asymptotic solution of problem (1.4)-(1.5) in the form of the sum of a regular part which approximates the solution at the internal points of the domains D_m and the boundary-layer parts which only play a substantial role in the subdomains, D_{mS} and $D_{m\Sigma}$ adjacent to the surfaces S_m and Σ.

Let us put

$$\varphi_m = \Phi_m \equiv \Phi_{m0} + R^{-1/2}\Phi_{m1} + \ldots \tag{2.1}$$

$$\mathbf{v}_m = S\mathbf{v}_m + \Sigma\mathbf{v}_m \equiv S_0\mathbf{v}_m + R^{-1/2}S_1\mathbf{v}_m + \ldots + \Sigma_0\mathbf{v}_m + R^{-1/2}\Sigma_1\mathbf{v}_m$$

Here Φ_m is the regular part of the asymptotic expansion while $S\mathbf{v}_m$ and $\Sigma\mathbf{v}_m$ are the boundary-layer terms which are only substantial in D_{mS} and $D_{m\Sigma}$ respectively.

We now introduce the curvilinear orthogonal coordinates s_1, s_2 and s into D_{mS} such that the surface $s = 0$ coincides with the surfaces S_m and $s > 0$ in D_m. Into $D_{m\Sigma}$, we introduce the rectangular Cartesian coordinates x, y_m and z_m such that the x-axis coincides with the axis of the initial coordinate system, $y_m \in \Sigma$ and the z_m-axes are directed into the domain D_m.

We treat the boundary-layer $S\mathbf{v}_m$ and $\Sigma\mathbf{v}_m$ terms as functions of s_1, s_2 and σ and x, y_m and ζ_m respectively where $\sigma = R^{1/2}s$, $\zeta_m = R^{1/2}z_m$ are "extended" coordinates. We require that the boundary-layer functions should satisfy the conditions

$$S\mathbf{v}_m \to 0 \quad \text{when} \quad \sigma \to \infty;\ \Sigma\mathbf{v}_m \to 0 \quad \text{when} \quad \zeta_m \to \infty \tag{2.2}$$

Let us substitute the expansion (2.1) into the system of Eqs.(1.4). We obtain separate equations for the regular and boundary-layer terms and we assume that the functions Φ_m as well as the Lamé parameters of the curvilinear coordinate systems which have been introduced vary slowly in the boundary layers D_{mS} and $D_{m\Sigma}$ as the coordinates s and Z_m vary compared with the function $S\mathbf{v}_m$ and $\Sigma\mathbf{v}_m$. We shall therefore subsequently consider Φ_m as functions of the initial coordinates x, y, z and the Lame parameters H_l ($l = 1, 2, 3$) as functions of s_1, s_2 and s, that is, we shall not introduce the arguments of the "extended" variables σ and ζ_m into them.

We will now substitute expansions (2.1) into the boundary conditions (1.5) and obtain relationships on S_m and Σ which link the regular and boundary-value terms of the expansions.

3. *The zeroth approximation.* Let us construct the zeroth approximation, that is, the functions which satisfy problem (1.4), (1.5) with an accuracy up to $O(R^{-1/2})$.

By considering quantities of the order of $R^{1/2}$ in the last equation of (1.4) and quantities of the order of R^0 in the second equation which has been projected onto the direction of the coordinate line s, we have

$$\frac{\partial}{\partial\sigma} S_0 v_{m3} = 0, \quad \frac{\partial}{\partial t} S_0 v_{m3} = \frac{\partial^2}{\partial\sigma^2} S_0 v_{m3}$$

By solving these equations allowing for the conditions (2.2) and, also, the analogous equations for $\Sigma_0 v_{m\zeta m}$, we find that

$$S_0 v_{m3} = \Sigma_0 v_{m\zeta m} = 0 \tag{3.1}$$

By considering quantities of the order of R^0 in the first equation of (1.4) and in those of the boundary conditions (1.5) which contains $S_0 v_{m3}$ and $\Sigma_0 v_{m\zeta m}$, we arrive at the following boundary-value problem:

$$\Delta\Phi_{m0} = 0 \quad \text{in} \quad D_m \tag{3.2}$$

$$\frac{\partial\Phi_{m0}}{\partial n} = 0 \quad \text{on} \quad S_m$$

$$\frac{\partial\Phi_{20}}{\partial z} - \frac{\partial\Phi_{10}}{\partial z} = 0, \quad \frac{\partial\Phi_{20}}{\partial z} - \rho\frac{\partial\Phi_{10}}{\partial z} + F\frac{\partial^2}{\partial t^2}(\Phi_{20} - \rho\Phi_{10}) = 0 \quad \text{on} \quad \Sigma$$

Here and below, $\mathbf{n}$ is the normal to the bounding surface of a domain D_m which is internal with respect to this domain.

The functions

$$\Phi_{m0}(M, t) = Cf_{m0}(M)\cos\psi(t) \tag{3.3}$$

$$(dC/dt = 0, \; d\psi/dt = \bar\omega, \; \bar\omega = \omega/\omega_0)$$

are particular solutions of the problem.

Here, $f_{m0}(M)$ is an eigenfunction of the boundary-value problem which is obtained from (3.2) after the operation of differentiation with respect to t has been replaced in the second boundary condition on Σ by multiplication by $-\bar\omega^2$, where the product $-F\bar\omega^2$ is equal to one of the eigenvalues of problem (3.2).

We note that problem (3.2) has a discrete spectrum of eigenvalues and, moreover, they are of finite multiplicity. We denote by $f_{m0}(M)$ one of the eigenfunctions which corresponds to the selected eigenvalue.

The solutions (3.3) describe the natural oscillations of an ideal fluid which have a constant amplitude C and a phase $\psi(t)$ which varies uniformly with time.

In treating the natural oscillations of a viscous fluid, we select the functions $\Phi_{m0}(M, t)$ in the form of (3.3) and, following the idea behind the method of averaging, we assume that the amplitude of the oscillations C and the rate of change of the phase $d\psi/dt$ vary slowly with time depending on the magnitude of the amplitude C itself and the phase difference $\theta = \psi - \bar\omega t$. Let us put

$$dC/dt = R^{-1/2}A_1(C, \theta) + R^{-1}A_2(C, \theta) + \ldots \tag{3.4}$$

$$d\psi/dt = \bar\omega + R^{-1/2}B_1(C, \theta) + R^{-1}B_2(C, \theta) + \ldots$$

where $A_1(C,\theta), B_1(C,\theta), A_2(C,\theta), \ldots$ are periodic functions of θ with a period of 2π which, like the expansion coefficients (2.1), are to be defined from problem (1.4), (1.5).

In calculating the partial derivatives with respect to time of the functions occurring in the second and third equations of (1.4) and the second conditions of (1.5), we assume that the functions are explicitly dependent on C and ψ and that the derivatives with respect to t are defined by formulae (3.4). For instance, the following expansions (we take account of relationships (2.1) and (3.4)) hold in the case of the derivatives of the functions Φ_m with respect to t:

$$\frac{\partial\Phi_m}{\partial t} = \bar\omega\frac{\partial\Phi_{m0}}{\partial\psi} + R^{-1/2}\left[\bar\omega\frac{\partial\Phi_{m1}}{\partial\psi} + \frac{\partial\Phi_{m0}}{\partial C}A_1 + \frac{\partial\Phi_{m0}}{\partial\psi}B_1\right] + \ldots \tag{3.5}$$

$$\frac{\partial^2\Phi_m}{\partial t^2} = \bar\omega^2\frac{\partial^2\Phi_{m0}}{\partial\psi^2} + R^{-1/2}\left[\bar\omega^2\frac{\partial^2\Phi_{m1}}{\partial\psi^2} + 2\bar\omega\left(\frac{\partial^2\Phi_{m0}}{\partial C\partial\psi}A_1 + \frac{\partial^2\Phi_{m0}}{\partial\psi^2}B_1\right)\right] + \ldots$$

Let us now find the functions $\Sigma_0 v_{mx}$, $\Sigma_0 v_{my_m}$, $S_0 v_{ml}$, where $l = 1, 2$.

The equations and boundary conditions for the functions Σ_0, which can be derived from the second equation of (1.4) and the second and third conditions of (1.5) using formulae (3.3) and (3.4) are:

$$\frac{\partial \Sigma_0 v_{m\xi_m}}{\partial \psi} = \bar{\omega}\mu_m \frac{\partial^2 \Sigma_0 v_{m\xi_m}}{\partial \zeta_m^2}, \quad \xi_m = x, y_m \quad \text{in} \quad D_{m\Sigma}$$
$$\rho\nu \frac{\partial \Sigma_0 v_{1\xi_1}}{\partial \zeta_1} \pm \frac{\partial \Sigma_0 v_{2\xi_2}}{\partial \zeta_2} = 0$$
$$\Sigma_0 v_{1\xi_1} \mp \Sigma_0 v_{2\xi_2} = \frac{\partial}{\partial \xi}(f_{10} - f_{20}) C \cos\psi \quad \text{on} \quad \Sigma$$

(the upper sign is taken when $\xi_1 = \xi_2 = \xi = x$ and the lower sign when $\xi_1 = y_1$, $\xi_2 = y_2$, $\xi = y$).

We note that $y_1 = -y_2 = y$ and that the functions f_{m0} are assumed to be dependent on the initial Cartesian coordinates x, y, z.

When account is taken of the requirement (2.2), the solutions of the latter equations have the form

$$\Sigma_0 v_{m\xi_m} = (\delta_{1m} \mp \rho\sqrt{\nu}\,\delta_{2m}) \frac{C}{1 + \rho\sqrt{\nu}} \frac{\partial}{\partial \xi}(f_{10} - f_{20})|_{z=0}\, e^{-\Omega\zeta_m} \cos(\psi - \Omega\zeta_m), \tag{3.6}$$
$$\Omega = \left(\frac{\bar{\omega}}{2\mu_m}\right)^{1/2}$$

the minus sign being taken when $\xi_m = x$, $\xi = x$ and the plus sign when $\xi_m = y_m$, $\xi = y$.

The functions $S_0 v_{ml}$, where $l = 1, 2$, which are found using the boundary conditions (1.5) on S_m, are

$$S_0 v_{ml} = H_l^{-1} \partial f_{m0}/\partial s_l\,|_{S_m}\, C e^{-\Omega\sigma} \cos(\psi - \Omega\sigma) \tag{3.7}$$

Here, H_1 and H_2 $(H_3 = 1)$ are the Lamé parameters of the coordinate system s_1, s_2, s.

By considering terms of the order of R^0 in the third equation of (1.4) and allowing for formulae (3.6) and conditions (2.2), we get

$$\Sigma_1 v_{m\zeta_m} = (\delta_{1m} - \rho\delta_{2m}) (\tfrac{1}{2}\nu/\bar{\omega})^{1/2} E e^{-\Omega\zeta_m} Q(\psi - \Omega\zeta_m) \tag{3.8}$$

$$S_1 v_{m3} = (\tfrac{1}{2}\mu_m/\bar{\omega})^{1/2} G_m C e^{-\Omega\sigma} Q(\psi - \Omega\sigma)$$
$$Q(\alpha) = \cos\alpha + \sin\alpha, \quad E = (\partial^2/\partial x^2 + \partial^2/\partial y^2)(f_{10} - f_{20})|_{z=0}$$
$$G_m(s_1, s_2) = \frac{1}{H_1 H_2}\left[\frac{\partial}{\partial s_1}\left(\frac{H_2}{H_1}\frac{\partial f_{m0}}{\partial s_1}\right) + \frac{\partial}{\partial s_2}\left(\frac{H_1}{H_2}\frac{\partial f_{m0}}{\partial s_2}\right)\right]\Bigg|_{S_m}$$

We note that the relationship

$$\rho\Sigma_1 v_{1\zeta_1} + \Sigma_1 v_{1\zeta_2}|_{\zeta_1=\zeta_2=0} = 0 \tag{3.9}$$

follows from the first equality of (3.8).

4. *Formulae for the damping coefficient and the correction to the frequency.* In order to derive formulae for the damping decrement and the correction to the frequency of the natural vibrations of an ideal fluid, we construct a number of first approximation equations in which problem (1.4), (1.5) is satisfied with an accuracy up to $O(R^{-1})$. It can be shown that, if terms of the order of $R^{-1/2}$ are equated in the second equation of (1.5), then the functions $\Sigma_1 v_{m\zeta_m}$, $S_1 v_{m3}$, which are defined by formulae (3.8), will satisfy the resulting equations.

The equations and boundary conditions for Φ_{m1} have the form

$$\Delta\Phi_{m1} = 0 \quad \text{in} \quad D_m \tag{4.1}$$
$$\partial\Phi_{m1}/\partial n = S_1 v_{m3} \quad \text{on} \quad S_m \tag{4.2}$$
$$\frac{\partial \Phi_{21}}{\partial z} - \frac{\partial \Phi_{11}}{\partial z} = -\Sigma_1 v_{2\zeta_2} - \Sigma_1 v_{1\zeta_1}$$
$$\frac{\partial \Phi_{21}}{\partial z} - \rho\frac{\partial \Phi_{11}}{\partial z} + F\bar{\omega}^2 \frac{\partial^2}{\partial \psi^2}(\Phi_{21} - \rho\Phi_{11}) = -2\bar{\omega}F\left[\left(\frac{\partial^2 \Phi_{20}}{\partial C\partial\psi} - \rho\frac{\partial^2 \Phi_{10}}{\partial C\partial\psi}\right)A_1 + \right.$$
$$\left.\left(\frac{\partial^2 \Phi_{20}}{\partial \psi^2} - \rho\frac{\partial^2 \Phi_{10}}{\partial \psi^2}\right)B_1\right] - \rho\Sigma_1 v_{1\zeta_1} - \Sigma_1 v_{2\zeta_2} \quad \text{on} \quad \Sigma \tag{4.3}$$

By reasons of arguments which are presented below, the functions $\Sigma_0 v_{mx}$, $\Sigma_0 v_{my_m}$ and $S_0 v_{m3}$

are omitted on the right-hand sides of the equalities (4.2) and (4.3) respectively.

According to relationships (3.3) and (3.8), the right-hand sides of (4.2) and (4.3) depend on ψ according to the signs of $\sin\psi$ and $\cos\psi$. It may therefore be assumed that the solutions Φ_{m1} of problem (4.1)-(4.3) satisfy the relationship $\partial^2\Phi_{m1}/\partial\psi^2 = -\Phi_{m1}$. When this is taken into account and, also, the representation (3.3) and relationship (3.9), we can write the boundary condition on Σ as:

$$\begin{aligned} &\partial\Phi_{21}/\partial z - \partial\Phi_{11}/\partial z = -\Sigma_1 v_{2\zeta_2} - \Sigma_1 v_{1\zeta_1} \\ &\partial\Phi_{21}/\partial z - \rho\partial\Phi_{11}/\partial z - F\varpi^2(\Phi_{21} - \rho\Phi_{11}) = \\ &2\varpi F(f_{20} - \rho f_{10})(A_1 \sin\psi + B_1 C\cos\psi) \quad \text{on} \quad \Sigma \end{aligned} \tag{4.4}$$

The parameter $F\varpi^2$, selected in paragraph 3, in equal to the characteristic number for problem (4.1), (4.2) and (4.4). A solution of the inhomogeneous boundary-value problem therefore only exists when conditions are imposed on the right-hand sides of relationships (4.2) and (4.4). These conditions enable us to find the magnitudes of A_1 and B_1 which determine the rate of the slow change in the amplitude C and the phase ψ of the vibrations.

Let us derive these conditions making use of a procedure which is analogous to that used in /1/.

Assuming that the solution of the above-mentioned boundary-value problem, that is, the functions Φ_{m1} have been found, we write Green's formulae for the functions f_{m0} and Φ_{m1}:

$$\int_{S_m} f_{m0}\frac{\partial\Phi_{m1}}{\partial n}dS - (-1)^m\int_{\Sigma}\left(f_{m0}\frac{\partial\Phi_{m1}}{\partial z} - \Phi_{m1}\frac{\partial f_{m0}}{\partial z}\right)d\Sigma = 0$$

By substituting expressions for the derivatives of the functions f_{m0} and Φ_{m1}, determined from (3.2), (3.3), (4.2) and (4.4), we get

$$\begin{aligned} &(1-\rho)\int_{S_m} f_{m0}S_1 v_{m3}\,dS + \int_{\Sigma}[2\varpi F_m Q_1 - r_m F_m\varpi^2\Phi_{m1}f_{3-m,0} + \\ &r_m f_{m0}(\Sigma_1 v_{1\zeta_1} + \Sigma_1 v_{2\zeta_2} + F_m\varpi^2\Phi_{3-m,1})]\,d\Sigma = 0 \end{aligned}$$

$$\begin{aligned} Q_1 = (A_1\sin\psi + CB_1\cos\psi)(f_{20} - \rho f_{10}),\ r_1 = 1,\ r_2 = -\rho,\ F_1 = \\ = F,\ F_2 = -F \end{aligned}$$

We note that, if the functions $\Sigma_0 v_{mx}, \Sigma_0 v_{my_m}$ had not been omitted from the right-hand sides of formulae (4.2) and (4.3), integrals having an order of $R^{-1/2}$ would have occurred on the left-hand sides of the latter equalities which equally would have had to be discarded in this approximation.

By multiplying the latter relationship, in which we have chosen $m = 1$, by ρ and summing it with the relationship in which $m = 2$, we obtain, after some reduction using Eqs.(3.1) and (3.9)

$$\begin{aligned} &2(1-\rho)F^{-1}\varpi^{-3}(A_1\sin\psi + CB_1\cos\psi)\int_{\Sigma}(\partial f_{20}/\partial z)^2\,d\Sigma = \\ &\rho\int_{S_1+\Sigma} f_{10}\mathbf{a}_1\mathbf{n}_1\,dS + \int_{S_2+\Sigma} f_{20}\mathbf{a}_2\mathbf{n}_2\,dS \\ &\mathbf{a}_m = S_0\mathbf{v}_m + R^{-1/2}S_1\mathbf{v}_m + \Sigma_0\mathbf{v}_m + R^{-1/2}\Sigma_1\mathbf{v}_m \end{aligned} \tag{4.5}$$

The components of the vectors $S_0\mathbf{v}_1, \Sigma_0\mathbf{v}_1, \ldots, \Sigma_1\mathbf{v}_2$ are defined by the formulae (3.1), (3.6)-(3.8) and, moreover, it is assumed that the components of the vectors $S_1\mathbf{v}_m$ and $\Sigma_1\mathbf{v}_m$, which are tangents to the surfaces S_m and Σ respectively, are equal to zero. We denote by $\mathbf{n}_1$ and $\mathbf{n}_2$ the unit normals to the boundaries of the domains D_m and Σ which are internal with respect to these domains.

We transform the right-hand side of (4.5), having made use of Gauss's theorem and taking account of the fact that the vectors $\mathbf{a}_m$ can be considered as being solenoidal in D_m (in fact, $\mathbf{a}_m$ are solenoidal with an accuracy up to $O(R^{-1/2})$ in $D_{mS}, D_{m\Sigma}$ and $|\mathbf{a}_m|$ decay rapidly as the point of integration becomes more remote from the boundaries of D_m). We get

$$2(1-\rho)F^{-1}\varpi^{-3}(A_1\sin\psi + CB_1\cos\psi)\int_{\Sigma}(\partial f_{20}/\partial z)^2\,d\Sigma = -\rho\int_{D_1}\nabla f_{10}\mathbf{a}_1\,dv - \int_{D_2}\nabla f_{20}\mathbf{a}_2\,dv \tag{4.6}$$

The integrals over the subdomains D_{mS} and $D_{m\Sigma}$ make the main contribution to the right-hand side of (4.6). In these subdomains, the derivatives of the functions f_{m0} with respect to s and z_m can be taken as being equal to zero with an accuracy up to $O(R^{-1/2})$, while the derivatives with respect to s_1, s_2 and x, y_m respectively interchange their values when $s = z_m = 0$. Allowing for this we integrate, in the volume integrals of (4.6), with respect to s and z_m within the limits from 0 to ∞ as a result of which the integrals are reduced to surface integrals and the equality (4.6) takes the form

$$2^{3/2}(1-\rho)F^{-1}\varpi^{-1/2}(A_1\sin\psi + CB_1\cos\psi) = -CI(\cos\psi+\sin\psi) \tag{4.7}$$

$$I == I_3^{-1}\left(\rho\sqrt{\nu}\,I_1 + I_2 + \frac{\rho\sqrt{\nu}}{1+\rho\sqrt{\nu}}I_4\right)$$

$$I_1 = \int_{S_1}(\nabla_2 f_{10})^2\,dS,\quad I_2 = \int_{S_2}(\nabla_2 f_{20})^2\,dS$$

$$I_3 = \int_{\Sigma}(\partial f_{20}/\partial z)^2\,d\Sigma,\quad I_4 = \int_{\Sigma}(\nabla_2 f_{10} - \nabla_2 f_{20})^2\,d\Sigma$$

Whence

$$A_1 = -CF2^{-3/2}\varpi^{1/2}(1-\rho)^{-1}I,\ B_1 = A_1/C$$

By substituting the values of A_1 and B_1 which have been found into Eqs.(3.4) and introducing dimensional variables once again using formulae (1.2), we arrive at the following expressions for the dimensional damping coefficient and the correction to the frequency:

$$\alpha = A_1\omega_0 C^{-1}R^{-1/2} = -(2^{-3}\nu_2\omega^5)^{1/2}\quad[g(1-\rho)]^{-1}I \tag{4.8}$$

$$\Delta\omega = B_1\omega_0 R^{-1/2} = \alpha$$

The quantities containing I_3 and I_4 determine the mechanical energy losses on the boundary of separation between the fluids, while the quantities containing I_1 and I_2 determine the energy losses on the walls of the vessel. In the general case, all these quantities are of the same order of smallness $R^{-1/2}$. In the specific case of a homogeneous fluid when $\rho = 0$, the energy losses on the free surface have a higher order of smallness which follows from (4.7) and (4.8) and from formula (2.21) in /1/ to which (4.8) reduces when $\rho = 0$.

5. *Examples.* Let us consider a vessel which has a vertical lateral surface, a planar bottom and a planar upper cover and select the function $f_{m0}(M)$ in the form (h_m are the depths of the fluid layers)

$$f_{m0} = (-1)^{m+1}\,\mathrm{sh}^{-1}\,\varkappa_n h_m\,\chi_n(x,y)\,\mathrm{ch}\{\varkappa_n[h_m + (-1)^m z]\}$$

where $\chi_n(x,y)$ is one of the eigenfunctions of the problem

$$\Delta_2\chi_n + \varkappa_n^2\chi_n = 0\quad \text{in}\quad \Sigma,\ \partial\chi_n/\partial n = 0\quad \text{on}\quad \Gamma \tag{5.1}$$

and Δ_2 is the Laplacian operator with respect to the variables x and y and Γ is the curve along which the plane Σ $(z=0)$ intersects the lateral surface of the vessel. The frequency ω_n of the vibrations of an ideal fluid is expressed in terms of $\varkappa_n$ using the formula

$$\omega_n^2 = g\varkappa_n(\rho_2-\rho_1)(\rho_2\,\mathrm{cth}\,\varkappa_n h_2 + \rho_1\,\mathrm{cth}\,\varkappa_n h_1)^{-1} \tag{5.2}$$

By substituting the functions f_{m0} into (4.7) and integrating with respect to z, we get

$$I_m = \mathrm{sh}^{-2}\varkappa_n h_m\left\{\frac{\mathrm{sh}\,2\varkappa_n h_m}{4\varkappa_n}\int_\Gamma\left[\left(\frac{\partial\chi_n}{\partial l_h}\right)^2 + \varkappa_n^2\chi_n^2\right]d\gamma + \right. \tag{5.3}$$

$$\left.\frac{h_m}{2}\int_\Gamma\left[\left(\frac{\partial\chi_n}{\partial l_h}\right)^2 - \varkappa_n^2\chi_n^2\right]d\gamma + \int_\Sigma\left[\left(\frac{\partial\chi_n}{\partial x}\right)^2 + \left(\frac{\partial\chi_n}{\partial y}\right)^2\right]d\Sigma\right\},\quad m=1,2$$

$$I_3 = \varkappa_n^2\int_\Sigma\chi_n^2\,d\Sigma,\quad I_4 = (\mathrm{cth}\,\varkappa_n h_1 + \mathrm{cth}\,\varkappa_n h_2)^2\int_\Sigma\left[\left(\frac{\partial\chi_n}{\partial x}\right)^2 + \left(\frac{\partial\chi_n}{\partial y}\right)^2\right]d\Sigma$$

Here, $\partial/\partial l_h$ are the derivatives with respect to the tangent to the curve Γ in the Σ plane.

In order to calculate the damping coefficient and the correction to the frequency, it is necessary to find the characteristic numbers $\varkappa_n$ and the eigenfunctions χ_n of problem (5.1) and, then, uisng formulae (5.3), to calculate I_m $(m=1,2,3,4)$ and to substitute the I_m into (4.8) and, at the same time, to adopt ω_n defined by the equality (5.2) as ω.

We will present the final expressions for I_m $(m=1,2,3,4)$ for two cases.

A. The vessel has the form of a rectangular parallelepiped $0\leqslant x\leqslant a$, $0\leqslant y\leqslant b$, $-h_2\leqslant z\leqslant h_1$. In this case

$$\varkappa_{sr}=\pi\left(\frac{r^2}{a^2}+\frac{s^2}{b^2}\right)^{1/2},\quad \chi_{sr}=\cos\frac{\pi rx}{a}\cos\frac{\pi sy}{b},\quad s=0,1,\ldots,\quad r=1,2,\ldots$$

$$I_m=\frac{1}{\operatorname{sh}^2\varkappa_{sr}h_m}\frac{ab}{2(2-\delta_{s0})}\left\{\frac{q}{\varkappa_{sr}}\left[\frac{r^2}{a^2}\left(\frac{2-\delta_{s0}}{b}+\frac{1}{a}\right)+\frac{s^2}{b^2}\left(\frac{2}{a}+\frac{1}{b}\right)\right]-\right.$$

$$\left.2h_m\pi^2\left(\frac{r^2}{a^3}+\frac{s^2}{b^3}\right)+\varkappa_{sr}{}^2\right\},\quad m=1,2\quad q=\pi^2\operatorname{sh}2\varkappa_n h_m$$

$$I_3=2^{-2+\delta_{s0}}\varkappa_{sr}{}^2ab,\quad I_4=(\operatorname{cth}\varkappa_{sr}h_1+\operatorname{cth}\varkappa_{sr}h_2)^2I_3$$

B. The vessel has the form of a circular cylinder $0\leqslant R^*\leqslant d,\ 0\leqslant\varphi<2\pi,\ -h_2\leqslant z\leqslant h_1$. In this case $\varkappa_{sr}=\nu_{sr}/d,\ s,\ r=1,2,\ldots,$ where ν_{sr} are the successive positive zeros of the derivatives of the Bessel functions $(J_\nu'(\nu_{sr})=0,\ 0<\nu_{s1}<\nu_{s2}<\ldots)$

$$\chi_{0r}=2^{-1/2}J_0(\nu_{0r}R^*/d),\quad \chi_{sr}^{(1)}=J_s(\nu_{sr}R^*/d)\cos s\varphi,\quad \chi_{sr}^{(2)}=J_s(\nu_{sr}R^*/d)\sin s\varphi$$

$$I_m=\frac{\pi J_s{}^2(\nu_{sr})}{2\operatorname{sh}^2\varkappa_{sr}h_m}\left[\frac{\operatorname{sh}2\varkappa_{sr}h_m}{2\nu_{sr}}(s^2+\nu_{sr}^2)+(\nu_{sr}^2-s^2)\left(1-\frac{h_m}{d}\right)\right],\quad m=1,2$$

$$I_3=\pi2^{-1}(\nu_{sr}^2-s^2)J_s{}^2(\nu_{sr}),\quad I_4=(\operatorname{cth}\varkappa_{sr}h_1+\operatorname{cth}\varkappa_{sr}h_2)^2I_3$$

REFERENCES

1. CHERNOUS'KO F.L., On the free vibrations of a viscous fluid in a vessel, Prikl. Mat. i Mekhan., 30, 6, 1966.
2. BUTUZOV V.F., NEFEDOV N.N. and FEDOTOVA E.V., Asymptotic solution of the linearized problem on the propagation of sound in a bounded medium with a low viscosity. Zh. vychisl. Mat. mat. Fiz., 27, 2, 1987.
3. HARRISON W.J., The influence of viscosity on the oscillations of superimposed fluids, Proc. London Math. Soc., 6, 1908.
4. THORPE S.A., On standing internal gravity waves of finite amplitude, J. Fluid Mech., 32, 3, 1968.
5. HOWES F.A., A model for viscous attenuation of waves on the common boundary of two superimposed fluids, Wave Motion, 9, 5, 1987.

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